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Perturbation theory for Rankine vortices

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A perturbation scheme is constructed to describe the evolution of stable, localized Rankine-type hydrodynamic vortices under the action of disturbances such as density stratification. It is based on the elimination of singularities in perturbations by using the necessary orthogonality conditions which determine the vortex motion. Along with the discrete-spectrum modes of the linearized problem which can be kept finite by imposing the orthogonality conditions, the continuous-spectrum perturbations play a crucial role. It is shown that in a stratified fluid, a single (monopole) vortex can be destroyed due to the latter modes before it drifts very far, whereas a vortex pair preserves its stability for a longer time. The motion of the latter is studied in two cases: smooth stratification and a density jump. For the motion of a pair under a small angle to the interface, a complete description is given in the framework of our theory, including the effect of reflection of the pair from a region with slightly larger density.

1. Introduction

The localized vortex structures in an incompressible fluid are among the traditional objects of fluid dynamics. Such vortices are significant due not only to their non-trivial individual dynamics but also due to peculiarities of their collective behaviour. Geophysical motions such as thermals or cyclones in the atmosphere and synoptic eddies in the ocean are some important examples. Discussion of various aspects of vortex dynamics can be found in basic literature such as Lamb (1932), Batchelor (1967), and Turner (1973). The most thoroughly developed area is the description of two-dimensional flows with idealized objects, or point vortices. In this area, rather general mathematical statements can be formulated (concerning, e.g. the integrability of the equations for point vortices). A variety of specific problems have been solved including the classical Kármán sheet (Lamb 1932), chaotic dynamics of two vortex dipoles (Manakov & Shchur 1983), surface wave generation by a moving vortex pair (Kurgansky 1986), etc. More complicated are problems in which the finite-size eddies serve as elementary objects. In the last few decades, some analytical (Melander, Styezek & Zabusky 1972; Abrashkin 1987) and mostly computational (e.g. Deem & Zabusky 1978) results have been obtained to demonstrate non-trivial patterns of vortex motion and disintegration in the process of interaction with other vortices and vortex sheets.

The theory of vortex motions in stratified fluids is much less complete. The analytical result here is associated almost exclusively with a phenomenological model suggested by Saffman (1979); other works are based on numerical or laboratory modelling (e.g. Hill 1975 and Dahm, Sheil & Tryggvason 1989). According to Saffman (1979), a Lamb vortex pair in a smoothly stratified fluid may, in a limited time interval, oscillate due to buoyancy. Hill (1975) has shown that at later stages, the pair is destroyed due to a vortex sheet which is formed at its boundary and shed in a stratified fluid.

Here, an approximate analytical theory is constructed which describes the motion of vortices interacting with each other and with the vorticity created by density stratification in a non-gravitating fluid. The idea of the method is somewhat similar to that of the direct perturbation theory for solitons (applicable to both integrable and non-integrable systems), in which a soliton preserves its integrity in the zero-order approximation and can radiate in higher-order approximations (see, e.g. Keener & McLaughlin 1979; Grimshaw 1979; and Gorshkov & Ostrovsky 1981; the 'ideology' of the scheme considered below is closer to that of the last paper). We also consider the motion of each vortex as a whole in the first-order approximation, which suggests that the perturbations are locally weak and non-destructive, although they may affect considerably the vortex motion during an extended time interval. For vortices such an approach should be applied more cautiously than for solitons because the latter are the wave formations characterized by a finite number of coupled parameters (velocity, amplitude, etc.), and there is always the possibility for radiation to carry the perturbations to infinity rather than them accumulating in the vicinity of the basic wave. The vortices have a looser structure and do not necessarily produce wave motions. As we shall see below, a single monopole eddy tends to accumulate perturbations which results in their eventual destruction, whereas for a vortex dipole the perturbations streamline a dipole and are carried out to infinity, leaving the vortex structure only slightly perturbed. In a number of important cases, a vortex can preserve its structure for a long enough time under the action of a perturbation, and in addition the method presented here can evaluate the conditions and describe the initial stage of the vortex destruction. After describing the general theory, we shall illustrate it by examples concerning the motion of finite-radii vortices and vortex pairs both in homogeneous and density-stratified fluids.

2. Perturbation theory for a single vortex

2.1. *General scheme*

Consider the well-known equation for the stream function $\psi(\boldsymbol{r}, t)$ describing two-dimensional motions of an ideal incompressible fluid (see e.g. Batchelor 1967).

$$\partial_t \Delta\psi + J(\psi, \Delta\psi) = \mu \hat{H}(\psi, \boldsymbol{r}, t). \tag{1}$$

Here, $J(\alpha, \beta) = \partial_x \alpha \partial_y \beta - \partial_x \beta \partial_y \alpha$ is the Jacobian, $\boldsymbol{r}\{x, y\}$ is a two-dimensional vector, $\hat{H}$ is an operator describing perturbations, and μ is a small parameter. The perturbations can be associated with, for example, the influence of other vortices and with a weak density stratification. For $\mu = 0$ this equation has a solution corresponding to a stationary symmetric vortex with an arbitrary radial distribution of angular velocity $\psi = \Psi(r)$, where r is the distance from the vortex centre in the (x, y)-plane, with natural boundary conditions

$$\left.\frac{\partial \Psi}{\partial r}\right|_{r=0,\infty} = 0 \tag{2}$$

(no motion in the centre and at infinity). The integral vorticity $\Omega = \int_0^\infty r\Delta\Psi \, \mathrm{d}r$ is supposed to be non-zero and of the order of unity. For $\mu \neq 0$, we shall be solving an initial problem with $\psi(\boldsymbol{r},0) = \Psi(r)$. We seek a solution in the form of a series

$$\psi(\boldsymbol{r},t) = \Psi(|r|) + \sum_{n>0} \mu^n \psi^{(n)}(\boldsymbol{r},t), \tag{3}$$

where we use the variable $\boldsymbol{r} = \boldsymbol{r} - \boldsymbol{R}_0(t)$, with $\boldsymbol{R}_0(t) = (X_0(t), Y_0(t))$ defining the slowly changing position of the vortex centre (we retain the notation $r = |\boldsymbol{r}|$). The velocity of the latter is also represented as a series

$$\frac{\mathrm{d}\boldsymbol{R}_0}{\mathrm{d}t} = \sum_n \mu^n \boldsymbol{U}^{(n)}(t). \tag{4}$$

Substituting the series (3) into equation (1), in each order of μ we obtain a linear equation,

$$\hat{L}\psi^{(n)} = H^{(n)} \tag{5}$$

with the initial condition $\psi^{(n)}(t=0) = 0$ and $\Delta\psi^{(n)}(t=0) = 0$. Here, $\hat{L}$ is a linear operator given by

$$\hat{L}\psi^{(n)} = \frac{\partial}{\partial t}\Delta\psi^{(n)} + J\left(\Psi, \Delta\psi^{(n)}\right) + J\left(\psi^{(n)}, \Delta\Psi\right), \tag{6}$$

and $H^{(n)}$ depends on the solutions in previous approximations. In particular, in the first-order approximation,

$$H^{(1)} = \hat{H}[\Psi(r), \boldsymbol{r}, t] - \boldsymbol{U}^{(1)} \cdot \nabla\Delta\Psi. \tag{7}$$

Here and below all spatial derivatives are taken with respect to the new variable $\boldsymbol{r} = \boldsymbol{r} - \boldsymbol{R}_0$.

For the linear problem (5) we shall use the Laplace transform. Because of the symmetry of Ψ, it is convenient to represent the solution of (5) in polar coordinates r, φ:

$$\left.\begin{aligned} \psi^{(n)} &= \sum_m \mathrm{e}^{\mathrm{i}m\varphi} \int_\gamma \psi_m^{(n)}(r,\lambda)\mathrm{e}^{\lambda t}\,\mathrm{d}\lambda, \\ H^{(n)} &= \sum_m \mathrm{e}^{\mathrm{i}m\varphi} \int_\gamma H_m^{(n)}(r,\lambda)\mathrm{e}^{\lambda t}\,\mathrm{d}\lambda, \end{aligned}\right\} \tag{8}$$

where $m = 0, 1, 2, \ldots$ is the number of an azimuthal harmonic, and γ is a standard contour. This results in the following equation:

$$\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right) - \frac{m^2}{r^2} - \frac{\mathrm{i}m}{r}\frac{\mathrm{d}/\mathrm{d}r(\Delta\Psi)}{\lambda + \mathrm{i}\Lambda_m(r)}\right]\psi_m^{(n)}(r,\lambda) = H_m^{(n)}(r,\lambda)[\lambda + \mathrm{i}\Lambda_m(r)]^{-1} = \tilde{H}_m^{(n)}. \tag{9}$$

Here, $\Lambda_m(r) = (m/r)\mathrm{d}\Psi/\mathrm{d}r$. The solution of this equation can be written in quadratures:†

$$\psi_m^{(n)}(r,\lambda) = f_{1m}\left(c_{1m}^{(n)} + \int_a^r f_{2m}\tilde{H}_m^{(n)} w_m^{-1}\,\mathrm{d}r'\right) + f_{2m}\left(c_{2m}^{(n)} - \int_\infty^r f_{1m}\tilde{H}_m^{(n)} w_m^{-1}\mathrm{d}r'\right), \tag{10}$$

where $f_{1,2}^{(m)}$ is a pair of fundamental solutions of equation (9) with a zero right-hand

† We consider zero initial condition for perturbations, because it is supposed that the unperturbed vortex is stable, i.e. the solutions of (10) with zero right-hand part remain finite.

part; $c^{(n)}_{1,2m}$ are integration constants (independent of r); $w_m(r,\lambda) = f'_{1m}f_{2m} - f_{2m}f'_{1m}$ is the Wronskian, and a is an arbitrary but fixed point in the interval $(0,\infty)$. Note that the Wronskian has a form $w = D(\lambda)/r$, where D is some function of and only of λ. It is constructive to define the fundamental system in such a way that $f_{1m} \sim r^m$ when $r \to 0$, and $f_{2m} \sim r^{-m}$ when $r \to \infty$; these solutions are infinite for $r \to \infty$ and $r \to 0$, respectively. The character of these asymptotics can be readily seen from consideration of equation (9) without the right-hand part and also without the last term in the left-hand part. The latter decays rapidly at both these limits; the other two terms immediately give the asymptotics mentioned before.

Then we define the integration parameters c in (10) by imposing the conditions of finiteness of the functions $\psi^{(n)}_m$ for $r \to 0, \infty$, which gives

$$c^{(n)}_{1m} = -\int_a^\infty \frac{f_{2m}\tilde{H}^{(n)}_m}{w_m}\,\mathrm{d}r, \qquad c^{(n)}_{2m} = \int_a^0 \frac{f_{1m}\tilde{H}^{(n)}_m}{w_m}\,\mathrm{d}r. \tag{11}$$

As a result, we can write the solution in the form

$$\psi^{(n)}_m(r,\lambda) = \int_0^\infty G_m(r,r',\lambda)H^{(n)}_m(r',\lambda)\mathrm{d}r', \tag{12}$$

where G_m is the Green function, which has the following form:

$$G_m(r,r',\lambda) = \left\{\begin{array}{ll} f_{1m}(r',\lambda)f_{2m}(r,\lambda), & r' < r \\ f_{1m}(r,\lambda)f_{2m}(r',\lambda), & r' > r \end{array}\right\}[w_m(\lambda + \mathrm{i}\Lambda_m(r'))]^{-1}. \tag{13}$$

2.2. *Resonances in discrete and continuous spectra*

Let us discuss the temporal behaviour of the perturbations. The method of Green's function has been employed in related problems, for example, by Smith & Montgomery (1995). Smith & Rosenbluth (1990) found an algebraic instability for cylindrical configurations (plasma column and cylindrical vortex). In our case algebraic perturbation growth is also possible (see below), and the problem is to suppress such growth by making the vortex move properly. The overall temporal dependence of the perturbation component $\psi^{(n)}$ can be found by using the inverse Laplace transform and then summing all angular harmonics. Of primary interest here are the singularities of G_m as a function of the complex parameter λ. From (13) there exist at least two types of singularities. The first type corresponds to zeros of the Wronskian w which produce a discrete spectrum of localized eigenfunctions (modes) of the vortex. There is a countable number of such functions, λ_m, corresponding to the roots of the transcendental equation $w_m(\lambda_m, r) = 0$. The second type of singularities is associated with the condition $\lambda = -\mathrm{i}\Lambda_m = -(\mathrm{i}m/r)\partial_r\Psi$. These perturbations are also localized in space but have a continuous spectrum; actually, the value of Λ_m depends on r, so that for each λ (from some interval of its values) a radius exists for which a pole of the Green function is realized. To find these modes, a singular behaviour (logarithmic branching) of the functions $f_{1,2m}$ should be, in general, taken into account. Evidently, the temporal behaviour of the perturbation $\psi^{(n)}_m$ in time depends radically on whether the 'force' $H^{(n)}_m$ is in resonance with the modes corresponding to λ_m and/or Λ_m. In the absence of resonances, the perturbation remains small for all t, and the series (3) is adequate, hence the vortex flow is only weakly disturbed. Note that the 'discrete' part is localized in the vicinity of vortex cores, whereas the second is, in general, spread over larger scales.

On the other hand, resonances can yield a secular growth of $\psi^{(n)}_m$ in time, resulting in the eventual destruction of the initial vortex.† By analogy with the soliton theory, the cause of this can be related to the localization of perturbation modes so that disturbances are cumulated in the vortex area rather than radiated to infinity. In turn, the effects of resonances associated with the discrete- and continuous-spectrum perturbations are different. Let us start from the discrete-spectrum modes. It can be shown that, by allowing a slow dependence of the parameters of the zero-order solution Ψ on time and using the orthogonality (compatibility) conditions, one can suppress the resonances and describe a sufficiently long-time evolution of a vortex as a whole. As a most important case of resonance suppression in discrete-spectrum perturbations, we consider the translational mode with $m = 1$, which evidently corresponds to the displacement of a vortex as a whole. It has been known since the 1950s (Fjortoft 1950; Dikii 1960) that (9) always has a stable translational mode with $m = 1$, and that for $m = 1$ the problem for both discrete mode and continuous spectrum can be reduced to quadratures in a general case (Smith & Rosenbluth 1990).

In such a degenerate case, $\lambda_1 = 0$. Because, by definition of λ, $w_1(\lambda = 0, r) = 0$, then $f_{11}(\lambda_1 = 0, r) \sim f_{21}(\lambda = 0, r) \sim f_1(r) \sim \partial\Psi/\partial r$; the last expression can be readily verified directly by substitution into (9). Then from (13) with the expressions for Λ and w given above, it follows that $G_1(r, r', \lambda = 0) \sim f_1(r) r'^2$, and a translational discrete-spectrum part of the perturbation (13) has the form

$$\psi_1^{(n)}(r,t) \sim f_1(r) \int_0^\infty \mathrm{d}t' \int_0^\infty \mathrm{d}r' r'^2 H_1^{(n)}(r', t'),$$

where $H_1^{(n)} = \int_0^{2\pi} H^{(n)}(r,\varphi) \mathrm{e}^{\mathrm{i}\varphi}\, \mathrm{d}\varphi$ is the corresponding component of the perturbation $H^{(n)}$. In the complex plane ($r\mathrm{e}^{\mathrm{i}\varphi} = x + \mathrm{i}y$), we have

$$\psi_1^{(n)}(r,t) = f_1(r) \int_0^t \mathrm{d}t' \int_{-\infty}^\infty \int_{-\infty}^\infty (x' + \mathrm{i}y') H_1^{(n)}(x', y', t')\, \mathrm{d}x'\, \mathrm{d}y'. \tag{14}$$

In general, this function has a secular divergence at $t \to \infty$. To eliminate this divergence, one should use the variability of the vortex position $\boldsymbol{R}_0(t)$, by defining the functions $\boldsymbol{U}^{(n)}(t)$ in (4). Moreover, in our case, an ambiguity in the definition of $\boldsymbol{U}^{(n)}$ can be used not only to prevent secular growth of $\psi_1^{(n)}$ with time, but even to eliminate all higher-order translational perturbations by making the spatial integral, I, ($\int \mathrm{d}x\, \mathrm{d}y$) in (14) zero. From this condition, the functions $\boldsymbol{U}^{(n)}(t)$ in (4) can be defined in each successive order, and (4) becomes an explicit equation for translational vortex motion. In the lowest approximation, $n = 1$, the equation $I = 0$ together with equation (7) for $H^{(1)}$ and the expression for the integral vorticity, $\Omega = 2\pi \int_0^\infty r\Delta\Psi\, \mathrm{d}r$, yields

$$\frac{\mathrm{d}\boldsymbol{R}_0}{\mathrm{d}t} = \Omega^{-1} \int \boldsymbol{r}\hat{H}(\Psi, \boldsymbol{r}, t) \mathrm{d}\boldsymbol{r} \tag{15}$$

(here $\mathrm{d}\boldsymbol{r} = \mathrm{d}x\, \mathrm{d}y$). Hence, we have at least one necessary condition of finiteness of the perturbation, which gives the equation for the vortex drift as a whole. This is, in a sense, similar to the soliton motion under the action of perturbations. However, this condition is not always sufficient; to verify this, it is necessary to analyse the behaviour of the continuous-spectrum perturbations.

† A different problem is that of exponential stability of vortices with variable vorticity. Such instability may exist without any external perturbations. We restrict ourselves by consideration of vortices that are *exponentially stable*. It should be noted that Llewellyn Smith (1995) has shown that vortices with a non-zero circulation are stable with respect to the first-mode disturbances.

2.3. *Rankine vortex*

As previously mentioned, to calculate the continuous-spectrum modes, one must employ another singularity in the form of a logarithmic branching of functions $f_{1,2m}$ near the point $\Lambda^{(m)}$. To simplify the situation, in what follows we shall restrict ourselves by considering a Rankine vortex, i.e. a homogeneous vorticity patch (here of unit radius):

$$\Delta\Psi = \omega \begin{cases} 1, & r < 1 \\ 0, & r > 1 \end{cases} \tag{16}$$

($\omega = \Omega/\pi$) for which the continuous-spectrum singularity does not exist. In this case, the fundamental system $f_{1,2}$ has the form

$$\begin{aligned} f_{1m}(r,\lambda) &= \begin{cases} r^m, & r < 1 \\ a(\lambda)r^m + b(\lambda)r^{-m}, & r > 1, \end{cases} \\ f_{2m}(r,\lambda) &= \begin{cases} a(\lambda)r^{-m} + b(\lambda)r^m, & r < 1 \\ r^{-m}, & r > 1, \end{cases} \end{aligned} \tag{17}$$

where $a(\lambda) = 1 - b(\lambda) = (\lambda + \mathrm{i}\lambda_m)(\lambda + \mathrm{i}\Lambda_m(1))^{-1}$, $\Lambda_m(1) = \Lambda_m(r = 1)$, $\lambda_m = (\omega/2)(m-1)$. The Wronskian of these eigenfunctions is $w = 2ma(\lambda)r^{-1}$. The expressions for $a(\lambda)$ and $b(\lambda)$ were obtained by matching conditions at $r = 1$:

$$[f_{1,2m} = 0], [\partial_r f_{1,2m}] = 2b(\lambda) f_{1,2m}, \tag{18}$$

which follow from equation (9) with zero right-hand part, considering that $\partial_r \Delta\Psi = \omega\delta(r-1)$. Brackets in (18) denote the difference of corresponding functions at $r = 1-0$ and $r = 1+0$. Substituting (17) into (13), we obtain the following expression for G_m:

$$G_m(r, r', \lambda) = \frac{\omega G_{1m}(r, r')}{2(\lambda + \mathrm{i}\lambda_m)(\lambda + \mathrm{i}\Lambda_m(r'))} + \frac{G_{2m}(r, r')}{\lambda + \mathrm{i}\Lambda_m(r')}, \tag{19}$$

where

$$G_{1m} = \frac{r'}{2m} f_m(r) f_m(r'), \qquad G_{2m} = \frac{r'}{2m} \begin{cases} (r'/r)^m, & r' < r \\ (r/r')^m, & r' > r, \end{cases}$$

$$f_m(r) = \begin{cases} r^{-m}, & r < 1 \\ r^m, & r > 1. \end{cases}$$

Here f_m are discrete-spectrum eigenmodes. Now, applying the inverse Laplace transform, we obtain the perturbation $\psi_m^{(n)}$ in the time domain:

$$\begin{aligned} \psi_m^{(n)}(r,t) = \int_0^t \mathrm{d}t' \int \mathrm{d}r' \Bigg\{ & G_{2m} \exp\left[\mathrm{i}\Lambda_m(r')(t-t')\right] \\ & + \omega G_{1m} \frac{\exp\left[\mathrm{i}\Lambda_m(r')(t-t')\right] - \exp\left[\mathrm{i}\lambda_m(t-t')\right]}{2(\lambda_m - \Lambda_m(r'))} \Bigg\} H_m^{(n)}(r', t). \end{aligned} \tag{20}$$

It is worthwhile to briefly comment on the structure of expressions (19) and (20). The part of the solution (20) with G_{1m} corresponds to the discrete-spectrum modes in the form $C_m^{(n)} f_m(r)$. Their spatial structure does not depend on the 'force' $H_m^{(n)}$.

The motions corresponding to these modes are potential everywhere except for the boundary $r = 1$ of the basic vortex Ψ. The first, translational mode describes a displacement of the vortex as a whole, while the rest of the modes are responsible for the deformation of the vortex core. It should be emphasized that the amplitude functions $C_m^{(n)}(t)$ for these modes depend not only on the discrete-spectrum modes but also on the continuous-spectrum: indeed, a term $\exp\,[\mathrm{i}\Lambda^{(m)}(r')(t-t')]$ is present also in the discrete-spectrum part of (20). On the other hand, the term with G_{2m} is responsible for the continuous spectrum and it depends on the radial structure of $H_m^{(n)}$. To have an idea of the general structure of these perturbations, one can consider an initial problem with $\psi(t=0)=0$, and an initial condition for $\partial_r\Delta\Psi$, such that $H_m^{(n)}=\delta(r-1)$. It reproduces the spatial structure of the discrete-spectrum mode $f_m(r)$, so that the general solution for $\psi_m^{(n)}$, in this particular case, describes the temporal evolution of the mth discrete-spectrum mode. Hence, in general, it is impossible to strictly separate the perturbations of discrete and continuous spectra. As a result, the finiteness of $\psi^{(n)}$ in (20) does not guarantee the suppression of all resonances. We see that the problem of finding and suppressing resonances is, in general, complicated by the problem of separation of the perturbations into the discrete- and continuous-spectrum parts.† Fortunately, in the case of vortices with a piecewise homogeneous distribution of vorticity, it turns to be sufficient to analyse only the part with G_{2m} in (20). As it is easy to see, after summing the angular harmonics $\psi_m^{(n)}$ over m, the corresponding part of the perturbation, $\psi_v^{(n)}$, satisfies equation (5) with a simplified operator $\hat{L}$:

$$\partial_t v^{(n)} + J(\Psi, v^{(n)}) = H^{(n)}, \tag{21}$$

where $v^{(n)}(\boldsymbol{r},t) = \Delta\psi_v^{(n)}$ is the vorticity perturbation. Equation (21) has the form of an inhomogeneous transport equation for $v^{(n)}$. A fundamental system for an mth harmonic of the left-hand part of this equation is $f_{1m} = r^m, f_{2m} = r^{-m}$, so that the function $G_{2m}(r,r')$ in (19) combined from these functions and responsible for the continuous-spectrum modes of the general problem (see the expression after (19)), is also a Green function of (21). As a result, a general solution of equation (5) can be represented as a sum of the solution for $\psi_v^{(n)}$ following from (21), and a set of eigenmodes for the discrete-spectrum perturbations of the basic vortex:

$$\psi^{(n)}(\boldsymbol{r},t) = \psi_v^{(n)}(\boldsymbol{r},t) + \sum_m C_m^{(n)}(t) f_m(r) \mathrm{e}^{\mathrm{i}m\varphi}, \tag{22}$$

where the coefficients $C_m^{(n)}(t)$ are defined from (20) as

$$C_m^{(n)}(t) = \int_0^t \mathrm{d}t' \int_0^\infty \mathrm{d}r \frac{r\omega f_m(r)}{4m(\Lambda_m(r)-\lambda_m)} \left[\mathrm{e}^{\mathrm{i}\Lambda_m(r)(t-t')} - \mathrm{e}^{\mathrm{i}\lambda_m(t-t')}\right] H_m^{(n)}(r,t'). \tag{23}$$

This form of the solution has a rather clear meaning. Because the last term in (6) is zero everywhere except for the points of the vortex boundary, it is natural to solve (5) separately in the inner and outer regions of the vortex, and then match the solutions at $r = 1$. In fact, expression (22) realizes the match by use of a properly chosen combination of discrete-spectrum modes in the last term. In many cases, the use of equation (21) has an advantage over the general solution (20) in terms of the analysis of the solution structure and the search for resonances. Moreover,

† In a corresponding problem for solitons, structural separation of perturbations is realized automatically. In these cases perturbations belonging to discrete and continuous spectra are mutually orthogonal and have a radically different structure. Namely, the former perturbations are localized while the latter are not; they just describe the radiation wave field arising in any scattering act.

as will be seen below, the description based on (21) allows a simple generalization to the evolution of an arbitrary system of separated vortices. For this, however, it is constructive to preliminarily consider the evolution of a vortex system without any external perturbations. Because the spatial projections of the characteristics of equation (21) coincide with the circular streamlines of the basic vortex, it is clear that the solution for $v^{(n)}(\boldsymbol{r}, t)$ can be reduced to a set of discrete-spectrum modes only if the force is localized at the circular contour of the vortex core. Any other distribution of the function $H^{(n)}(\boldsymbol{r}, t)$ also generates continuum-spectrum perturbations which do not describe just the translation of the vortex core and its small deformations but may eventually destroy it. On the other hand, when $v^{(n)}$ remains finite, the amplitudes $C_m^{(n)}$ provide a correct description of the vortex evolution and all modes except for the translational (resonance) one remain small perturbations. In accordance with the comments made about the general solution (20) regarding the role of discrete- and continuous-spectrum modes, expression (23) for the discrete-mode amplitudes $C_m^{(n)}$ should be supplemented by terms corresponding to modes of a similar structure extracted from the solution v^n. To achieve this, we represent $H^{(n)}(\boldsymbol{r}, t)$ as a sum

$$H^{(n)}(\boldsymbol{r}, t) = \mathscr{H}^{(n)}(\boldsymbol{r}, t) + \delta(r-1)h^{(n)}(\boldsymbol{r}, t), \tag{24}$$

where $\mathscr{H}^{(n)}$ is a part regular at $r = 1$, and $h^{(n)}$ is a singular part generating the eigenmodes. As a result, instead of (23) we have

$$C_m^{(n)}(t) = \frac{1}{2m}\int_0^t \mathrm{d}t' \left\{ h_m^{(n)}(t')\mathrm{e}^{-\mathrm{i}\lambda_m(t-t')} \right. \\ \left. + \int_0^\infty \mathrm{d}r \frac{\omega r f_m(r)}{2(\Lambda_m(r)-\lambda_m)} \left[\mathrm{e}^{-\mathrm{i}\lambda_m(t-t')} - \mathrm{e}^{-\mathrm{i}\Lambda_m(r)(t-t')}\right] \mathscr{H}_m^{(n)}(r, t') \right\}. \tag{25}$$

Let us represent the amplitudes $C_m^{(n)}$ in (25) in terms of the distributed vorticity $v^{(n)}$. We express the function $\mathscr{H}_m^{(n)}$ using equation (21):

$$\mathscr{H}_m^{(n)} = \partial_t v_m^{(n)} + \mathrm{i}\Lambda_m v_m^{(n)}. \tag{26}$$

Substituting this into (25), integrating the terms with $\partial_t v_m^{(n)}$ by parts and using zero initial conditions for $v_m^{(n)}$, we obtain a linear oscillator-type equation for $C_m^{(n)}$

$$\frac{\mathrm{d}C_m^{(n)}}{\mathrm{d}t} + \mathrm{i}\lambda_m C_m^{(n)} = F_m^{(n)},$$

where

$$F_m^{(n)}(t) = \frac{1}{2m}\left[h_m^{(n)}(t) + \frac{\mathrm{i}\omega}{2}\int_0^\infty r f_m(r) v_m^{(n)}(r, t)\,\mathrm{d}r\right].$$

It yields a more compact expression for $C_m^{(n)}$ than (25):

$$C_m^{(n)} = \frac{1}{2m}\int_0^t \mathrm{d}t' \left\{ h_m^{(n)}(t') + \frac{\mathrm{i}\omega}{2}\int_0^\infty r f_m(r) v_m^{(n)}(r, t')\,\mathrm{d}r \right\} \mathrm{e}^{-\mathrm{i}\lambda_m(t-t')}. \tag{27}$$

2.4. *Equation for the motion of the vortex centre*

Here again, we have to eliminate the possible secular growth of $C_1^{(n)}(t)$ at large t, for which we let the expression in curly brackets go to zero. Then, introducing the slowly varying coordinates of the vortex centre as described above, and the slow dependence of $h_m^{(n)}$ on t following from (7), one can obtain, instead of (15), the following equation of motion:

$$\left(\boldsymbol{\Omega}\times\boldsymbol{U}^{(n)}\right)=\int \mathrm{d}\boldsymbol{r}\{\nabla\Psi v^{(n)}(\boldsymbol{r},t)+\Omega(\boldsymbol{\Omega}\times\boldsymbol{r})\delta(r-1)h^{(n)}(\boldsymbol{r},t)\}, \tag{28}$$

where $\boldsymbol{\Omega}=\Omega\boldsymbol{z}$.

The values of the velocity found from (15) and (28) coincide only if, in (24), $\mathscr{H}^{(n)}\equiv 0$ and, consequently, $v_m^{(n)}=0$; in general, however, only (28) yields a correct result because it contains the singular continuous-spectrum modes as well. It is worth noting that the point-vortex approximation corresponds to the case when in the integrand of (28), the contribution of the region occupied by the vortex core and its boundary can be neglected. As a result, (28) is reduced to the expression for the velocity induced by the vorticity perturbations v in the centre of the vortex. Hence, for the Rankine vortex, the problem can be solved completely.†

It is important for further analysis to examine how to extend the representation (22) together with (15) and (28) to the description of a vortex ensemble, provided the eddies are sufficiently separated from each other. This possibility will be demonstrated below.

3. Vortex system in a homogeneous fluid

To illustrate the theory for several vortices, let us start with the known problem of the motion of a finite-size vortex ensemble in a homogeneous fluid (a two-vortex configuration is illustrated in figure 1). We suppose that at the initial moment, the vortices have finite radii a_i which are small compared to the distances R_{ij} between their centres, so that the ratio a_i/R_{ij} is of the order of a small parameter μ. The vorticities ω_i are given to be constant for each vortex as above. The evolution of the flow is then reduced to the drifts and deformations of vortex cores, whereas the motion between them remains potential. It is constructive to consider this case first because it enables comparison of the results with those obtained before. As follows from the previous section, the motion in question is described by discrete-spectrum perturbations alone. Consequently, it is natural to represent the solution in the form of a series similar to (3), where the zero-order approximation corresponds to the superposition of unperturbed eddies and to the perturbations expanded in a series of discrete-spectrum modes:

$$\psi(\boldsymbol{r},t)=\sum_i \Psi(r_i)+\sum_n \mu^n\psi^{(n)}(\boldsymbol{r},t), \tag{29}$$

where $\boldsymbol{r}_i=\boldsymbol{r}-\boldsymbol{R}_i$, $r_i=|\boldsymbol{r}_i|$, and $\boldsymbol{R}_i=\{X_i,Y_i\}$ define the coordinates of the centre of the ith vortex. The perturbation $\psi^{(n)}$ is represented as follows:

$$\psi^{(n)}(\boldsymbol{r},t)=\sum_i \psi^{(n)}(\boldsymbol{r}_i,t)=\sum_{i,m} C_{mi}^{(n)}(t)f_m(r_i)\,\mathrm{e}^{\mathrm{i}m\varphi_i}. \tag{30}$$

Here φ_i is an angular variable in polar coordinates matched to the centre of the ith eddy. Let us substitute the series (29) into the basic equation (1) in which the right-hand part is set to be zero (no perturbations except for those from other vortices).

† Similarly to the soliton case, there can exist more parameters describing the basic vortex, which may create a need for additional compatibility conditions determining the vortex motion. As for the solitons, this problem must be considered in each specific case.

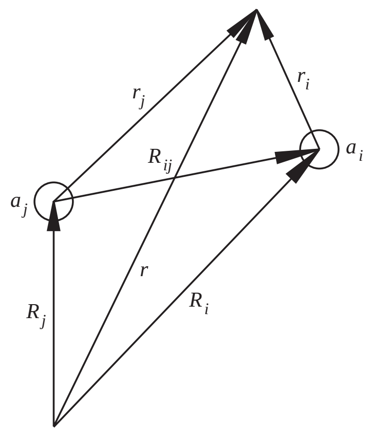

FIGURE 1. Geometry of a system of two vortices.

Then, in each order of μ we obtain a linear equation analogous to (5), (6)

$$\frac{\partial}{\partial t}\Delta\psi^{(n)} + J\left(\sum_i \Psi(r_i), \Delta\psi^{(n)}\right) + J\left(\psi^{(n)}, \Delta\sum_i \Psi(r_i)\right) = H^{(n)}. \tag{31}$$

The solution of this equation is supposed to be the sum of the modes described by the right-hand part of equation (30). Because the field of eigenmodes decreases as r^{-1} or faster, perturbations generated in the vicinity of the ith vortex are always of a higher order of smallness in the vicinities of all other vortices. Hence, in each approximation the perturbation eigenmodes can be found independently for each of the eddies. The forces $H^{(n)}$ which are due to the vortex interactions are also separated; moreover, they are strictly localized at the boundaries of vortex nuclei. Indeed, these functions consist of expressions such as $J(\psi^{(n-k)}, \Delta\psi^{(k)})$ in which $\Delta\psi^{(k)} = 0$ everywhere, with the exception of the points belonging to the contours of the vortices. As a result, the solution of equation (31) is equivalent to the solution of the corresponding problem for a single vortex considered above, whereas expressions (29) and (30) can be considered as expansions that are valid throughout all the (x, y)-plane. Let us now consider the temporal behaviour of the solution. Starting with the basic flow in the form of superposition of unperturbed eddies $\Psi(r_i, t)$,we obtain the following equation for the first-order perturbation $\psi^{(1)}(\boldsymbol{r}_i, t)$:

$$\hat{L}\psi^{(1)}(\boldsymbol{r}_i, t) = -\sum_{i\neq j} J(\Psi(r_j), \Delta\Psi(r_i)), \tag{32}$$

where $\hat{L}$ is the same operator as (6), with $\Psi = \Psi(r_i)$. The unperturbed flow around a single vortex is $\Psi = \omega a^2 \ln\left(r/a\right)$ so that the flow created by the jth vortex in the vicinity of the ith vortex, is

$$\Psi(r_j) = \Psi(|\boldsymbol{r}_i + \boldsymbol{R}_{ij}|) = \omega_j a_j^2 \ln\left|1 + \left(\frac{r_i}{R_{ij}}\right)^2 - 2\frac{r_i}{R_{ij}}\cos\left(\varphi_{ij} - \varphi_i\right)\right|, \tag{33}$$

where

$$R_{ij}^2 = (X_i - X_j)^2 + (Y_i - Y_j)^2, \qquad \tan\varphi_{ij} = \frac{Y_i - Y_j}{X_i - X_j}. \tag{34}$$

Expanding the function Ψ given by (33) in a series of r_i/R_{ij} and keeping only the first-order terms, after substitution into the right-hand part of (32), we obtain

$$\sum_{i\neq j} J(\Psi(r_j), \Delta\Psi(r_i)) = \sum_{i\neq j} \frac{1}{r_i}\frac{\partial\Psi(r_j)}{\partial\varphi_i}\frac{\partial\Delta\Psi(r_i)}{\partial r_i}$$

$$= \sum_{i\neq j} \frac{\omega_i\omega_j a_j^2}{R_{ij}}\delta(r_i - a_i)\sin(\varphi_i - \varphi_{ij}). \tag{35}$$

Substituting this into (32) and comparing with the singular term in the right-hand side of (24), one obtains $h_1^{(1)}$, which readily yields a term proportional to t in the expression (27) for $C_1^{(1)}$. Now, the corresponding solution for the first (translational) mode with $m = 1$ (which only exists in the first approximation here) follows from (22):

$$\psi_i^{(1)} = t\sum_{i\neq j} \frac{\omega_i\omega_j a_j^2}{R_{ij}} f_1(r)\exp[\mathrm{i}(\varphi_i - \varphi_{ij})], \tag{36}$$

which grows secularly in time. Actually, this describes an initial stage of the vortex drift as a whole. To eliminate this growth, we should again consider slow variations of the vortex coordinate by changing Ψ_i for $\Psi_i[X - X_i(t), Y - Y_i(t)]$. Then the right-hand part of (35) will contain an additional term in the form

$$-J\left(\dot X_i y - \dot Y_i x, \Delta\Psi_1\right) = \frac{1}{r_i}\frac{\partial}{\partial r_i}\Delta\Psi_1(r_i)\frac{\partial}{\partial\varphi_i}\left(\dot Y_i\cos\varphi_i - \dot X_i\sin\varphi_i\right). \tag{37}$$

In this particular case, the compatibility condition (that of finiteness of the first-order perturbation) simply means that the modified right-hand part of (35) must be zero, which gives

$$\dot X_i = \sum_{i\neq j}\frac{\omega_i a_j^2}{R_{ij}}\sin\varphi_{ij}, \qquad \dot Y_i = \sum_{i\neq j}\frac{\omega_i a_j^2}{R_{ij}}\cos\varphi_{ij}. \tag{38}$$

As expected, these are the classic equations for a system of point vortices. As a next step, one can find the perturbations $\psi^{(n)}$ in higher approximations. Remembering that the compatibility conditions yield $\psi^{(1)}(\boldsymbol{r}_i, t) = 0$, we obtain for the next-order perturbation $\psi^{(2)}$ an equation analogous to (32) in which only the terms of $O(a_j^2/R_{ij}^2)$ are kept in $\Psi(r_j)$:

$$\hat L\psi^{(2)}(\boldsymbol{r}_i, t) = -\sum_{i\neq j} J[\Psi(r_j), \Delta\Psi(r_i)]$$

$$= -r_i\omega_i\delta(r_i - a_i)\sum_{i\neq j}\frac{\omega_j a_j^2}{R_{ij}^2}\sin 2(\varphi - \varphi_{ij}). \tag{39}$$

This expression shows that only a second eigenmode of the vortex core oscillations is excited in the second approximation. This mode has a form $C_{2i}^{(2)}(t)f_2(r_i)\mathrm{e}^{2\mathrm{i}\varphi}$ and an amplitude

$$C_{2i}^{(2)}(t) = \tfrac{1}{2}\omega_i\omega_j a_j^2 \mathrm{e}^{-2\mathrm{i}\lambda t}\int_0^t R_{ij}^{-1}\exp\mathrm{i}\left[\lambda_{2i}t' - 2\varphi_{ij}(t')\right]\mathrm{d}t', \tag{40}$$

where $\varphi_{ij}(t')$ and $R_{ij}(t')$ are defined as solutions of equations (38). Because $\dot\varphi_{ij}/\varphi_{ij} \sim \dot R_{ij}/R_{ij} \sim \omega_i a_i/R_{ij}$, the functions φ_{ij} and R_{ij} are slowly varying compared to $\exp(\mathrm{i}\lambda_{2i}t)$, and the expression (40) describes slowly modulated, elliptic perturbations of vortex cores. In the particular case of a vortex pair when $R_{12}^2 = \text{const}$ and $\dot\varphi_{12} =$

$(a_1^2\omega_1 + a_2^2\omega_2)/R_{12}^2$, expression (40) coincides with an approximate solution obtained by Melander *et al.* (1972) and Abrashkin (1987), where different specific cases of this expression are discussed in detail.†

4. Vortex ensemble in the presence of perturbations

Now we consider the general case of perturbed equation (1). We shall use the results of the previous section concerning vortex systems in a homogeneous fluid. We suppose for the sake of simplicity that the order of perturbations ($\mu\hat{H}$) is, in general, the same as that for the vortex interactions, i.e. the small parameter μ is universal for all the effects considered. The principal difference of the general case from that considered in the previous section is that the flow between the vortex cores does not remain exactly potential. As a result, perturbations in the vicinity of each vortex cannot be described exclusively by the discrete-spectrum modes, and a more general representation (22) must be used which includes the component ψ_v with a non-zero vorticity. Unlike discrete-spectrum modes, this component does not necessarily decrease fast enough with distance from a given vortex. As a result one must match the perturbations $\psi_v^{(n)}$ found in the vicinities of different eddies in each approximation. Moreover, even for discrete-spectrum modes an ambiguity arises regarding an integration region because now these modes are not strictly localized at the vortex contours. In view of these problems, we shall construct a solution as a general series (3), with the zero-order solution in the form of a superposition of localized vortices, as in § 3:

$$\Psi(\boldsymbol{r}) = \sum_i \Psi_i(r_i). \tag{41}$$

The perturbations $\psi^{(n)}$ are supposed to be small with respect to all vortex flows Ψ_i, and they satisfy the linear equation

$$\frac{\partial}{\partial t}\Delta\psi^{(n)} + \sum_i \left[J\left(\Psi(r_i), \Delta\psi^{(n)}\right) + J\left(\psi^{(n)}, \Delta\Psi(r_i)\right)\right] = H^{(n)}. \tag{42}$$

For a selected jth vortex, this equation can be written in the form

$$\frac{\partial}{\partial t}\Delta\psi^{(n)} + J\left(\Psi(r_j), \Delta\psi^{(n)}\right) + J\left(\psi^{(n)}, \Delta\Psi(r_j)\right)$$
$$= H^{(n)} - \sum_{i\neq j} \left[J\left(\Psi(r_i), \Delta\psi^{(n)}\right) + J\left(\psi^{(n)}, \Delta\Psi(r_i)\right)\right] = \hat{H}^{(n)}. \tag{43}$$

Then we use the solution (20) obtained for a single vortex:

$$\psi_m^{(n)}(r,t) = \int_0^t \mathrm{d}t' \int \mathrm{d}r_j' \left\{ G_{2m}\left(r_j, r_j'\right) \exp\left[\mathrm{i}\Lambda^{(m)}\left(r_j'\right)\left(t-t'\right)\right] \right.$$
$$\left. + \omega_j G_{1m}\left(r_j, r_j'\right) \frac{\exp\left[\mathrm{i}\Lambda_m(r_j')(t-t\prime)\right] - \exp\left[\mathrm{i}\lambda_m(t-t')\right]}{2(\lambda_m - \Lambda_m(r_j'))} \right\} \hat{H}_m^{(n)}(r_j', t). \tag{44}$$

† In these papers which consider a vortex pair in similar conditions, the equations for vortex perturbations are nonlinear, unlike the solution given above. This difference is just due to a different choice of variables: instead of the complex amplitude $C_{2i}^{(2)}(t)$ used above, they operate with real amplitude and phase such as $C_{2i}^{(2)}(t) = A_i(t)\mathrm{e}^{\mathrm{i}\varphi_i(t)}$ $(i = 1, 2)$. Indeed, a linear equation $(\mathrm{d}/\mathrm{d}t + \mathrm{i}\lambda_{2i})C_{2i}^{(2)} = F_i\mathrm{e}^{\pm\mathrm{i}\varphi_{1,2}}$ for which (40) is a solution, will acquire a nonlinear form with respect to the variables A_i and φ_i.

This is essentially an integral equation ($\hat{H}_m^{(n)}$ depends on $\psi_m^{(n)}$). It is equivalent to (42), but this form is more helpful for obtaining an explicit solution. The term with G_{1m} evidently describes an mth azimuthal mode of discrete-spectrum perturbations which is proportional to $\phi_{mj}^{(n)} = C_{mj}(t) f_m(r_j) \mathrm{e}^{\mathrm{i}m\varphi_i}$. The part with G_{2m} which is equal, after summation over m, to $\tilde{\psi}^{(n)} = \psi^{(n)} - \Phi_j^{(n)}$ ($\Phi_j^{(n)} = \sum_m \phi_{mj}^{(n)}$), satisfies the equation

$$\frac{\partial}{\partial t}\Delta\tilde{\psi}^{(n)} + \sum_{i\neq j} J(\Psi(r_i), \Delta\tilde{\psi}^{(n)}) + \sum_{i\neq j} J\left(\tilde{\psi}^{(n)}, \Delta\Psi(r_i)\right)$$
$$= H^{(n)} - \sum_{i\neq j}\left[J\left(\Psi(r_i), \Delta\Phi_j^{(n)}\right) + J\left(\Phi_j^{(n)}, \Delta\Psi(r_i)\right)\right]. \quad (45)$$

This relation can be verified in the same way as for a single vortex (see above, § 2). In the right-hand part of (45) all terms with sums are of order μ^{n+1} or higher, and they can be omitted. These terms contain the products of values defined in the vicinities of different vortices, one of these values being strictly localized at the boundary of one of the vortices while the other is of order (at least) R_{ij}^{-1} with respect to the distance between vortices. As a result, we arrive at (42) in which a term with $i = j$ in the last sum is eliminated. This can be performed for each vortex which results in the following representation of the full solution:

$$\psi^{(n)}(\boldsymbol{r}, t) = \psi_v^{(n)}(\boldsymbol{r}, t) + \sum_{m,j} C_{mj}^{(n)}(t) f_m(r_j) \mathrm{e}^{\mathrm{i}m\varphi_i}, \quad (46)$$

which is a natural generalization of (22) for a multivortex case. The function $\psi_v^{(n)}$ satisfies an inhomogeneous transfer equation for the vorticity perturbation $v^{(n)} = \Delta\psi_v^{(n)}$ in the flow created by the basic eddies:

$$\partial_t v^{(n)} + J\left(\sum_i \Psi(r_i), v^{(n)}\right) = H^{(n)}. \quad (47)$$

By analogy with the case of a single vortex, this part can be ascribed to the continuous-spectrum part of the solution for the linear problem (42). As previously mentioned, the last sum in (46) is a superposition of the discrete-spectrum modes corresponding to separate eddies. According to (44), temporal evolution of their amplitudes is described by the expression

$$C_{mj}^{(n)}(t) = \frac{\omega_i}{4m}\int_0^t \mathrm{d}t' \int_0^\infty \mathrm{d}r_j f(r_j)$$
$$\times \frac{\exp\left[-\mathrm{i}\Lambda_m(r_j)(t-t')\right] - \exp\left[-\mathrm{i}\lambda_m(t-t')\right]}{\Lambda_m(r_j) - \lambda_m} \hat{H}_m^{(n)}(r_j, t). \quad (48)$$

Because $\hat{H}^{(n)}$ depends on $\psi^{(n)}$, it may seem that these modes are coupled via $\hat{H}_m^{(n)}$. In fact, however, the coupling can be neglected. Indeed, the same estimates as those made for equation (42), show that in the expression for $\hat{H}^{(n)}$ entering (48), the sum $\sum_{i\neq j} J\left(\psi^{(n)}, \Delta\Psi(r_i)\right)$ can be omitted, and $\Delta\psi^{(n)}$ can be replaced by $\Delta\psi_v^{(n)} = v^{(n)}$, i.e.

$$\hat{H}^{(n)} = H^{(n)} - J\left(\sum_{i\neq j} \Psi(r_i), v^{(n)}\right) + O(\mu^{n+1}). \quad (49)$$

A further simplification is possible. Using (47), we can express $\hat{H}^{(n)}$ in terms of $v^{(n)}$:

$$\hat{H}^{(n)} = \partial_t v^{(n)} + J\left(\Psi(r_j), v^{(n)}\right). \tag{50}$$

Integrating the term with $\partial_t v$ over t by parts in (48) with the initial condition $v^{(n)}(\boldsymbol{r}, t=0) = 0$, we finally have

$$C_{mj}^{(n)}(t) = \frac{\omega_j}{4m}\int_0^t \mathrm{d}t' \int_0^\infty \mathrm{d}r_j f(r_j) \exp\left[-\mathrm{i}\lambda_m(t-t')\right] v_m^{(n)}(r_j, t). \tag{51}$$

Hence, the solution of the linear problem (42) has been reduced to the solution of a linear first-order equation (47) for the vorticity perturbation $v^{(n)}$, and the subsequent quadratures (51) defining the discrete-spectrum modes. It is important that the form (46) for $\psi^{(n)}$ automatically solves the problem of matching for vortical flow components generated by different eddies. Moreover, it eliminates the aforementioned ambiguity in calculation of the amplitudes $C_{mj}^{(n)}$ because expressions (48) and (51) were obtained from the condition of smallness of $\psi^{(n)}$ with respect to the flow created by the whole system of basic eddies. In this sense, the representation (46) is a natural generalization of the homogeneously valid solution (30) obtained for a vortex ensemble without perturbations.

5. A single vortex near an interface

Now we consider some examples of vortex motions in inhomogeneous fluids. We begin with a ‘bad’ situation which is when perturbations destroy the vortex. The example is the evolution of a single vortex in a density-stratified fluid without gravity. We still take into account the density effect on the inertial properties of the fluid elements and, as a result, generation of vorticity in the fluid. The hydrodynamic equations for such a system have the form

$$\partial_t \Delta\psi + J(\psi, \Delta\psi) = -\left(\frac{\nabla\rho}{\rho}\frac{\mathrm{d}}{\mathrm{d}t}\psi\right), \tag{52}$$

$$\partial_t \rho + J(\psi, \rho) = 0, \tag{53}$$

where $\mathrm{d}/\mathrm{d}t = \partial/\partial t + (\boldsymbol{\nabla}\cdot\boldsymbol{v})$. Denoting by ρ_0 the unperturbed constant density, we suppose that $\delta\rho = \rho - \rho_0 \ll \rho_0$ everywhere. Then, the perturbation scheme developed above is applicable. In the zero-order approximation $\psi = \Psi(r)$, and $\rho = \rho_0 = \mathrm{const.}$ The first-order perturbation satisfies the equations

$$\partial_t \Delta\psi^{(1)} + J\left(\psi^{(1)}, \Delta\Psi\right) + J\left(\Psi, \Delta\psi^{(1)}\right) = -\left(\frac{\rho^{(1)}}{\rho}\frac{\mathrm{d}}{\mathrm{d}t}\Psi\right) = H^{(1)}, \tag{54}$$

$$\partial_t \rho^{(1)} + J\left(\Psi, \rho^{(1)}\right) = 0 \tag{55}$$

with the initial conditions

$$\psi^{(1)}(t=0) = 0, \qquad \rho^{(1)}(t=0) = \delta\rho(y). \tag{56}$$

Here, $\delta\rho(y)$ is a known function describing the initial stratification. Equation (55) is readily integrated in polar coordinates to give

$$\rho^{(1)} = \delta\rho\left[r\sin\left(\varphi - \frac{t}{r}\Psi_r\right)\right], \tag{57}$$

where, as above, polar coordinates r and φ are centred in the vortex centre. After this, the right-hand part of (54) can be written as

$$H^{(1)} = \left(\frac{\partial_r \Psi}{r}\right)^2 \cos\left(\varphi - \frac{t}{r}\partial_r \Psi\right)\frac{\rho^{(1)\prime}}{\rho_0}, \tag{58}$$

where $\rho^{(1)\prime}$ denotes the derivative with respect to the argument of $\delta\rho$ in (57). In the above formula (22) for perturbations, the distributed vorticity perturbation $v^{(1)}$ satisfies the inhomogeneous transfer equation (21) with the right-hand part of (58). From comparison of (21) with (55) or directly with the argument $\varphi - (t/r)\partial_r \Psi$ in (58) it is easily seen that the force (58) propagates along the characteristics of equation (21). Thus the latter has a resonance solution

$$v^{(1)}(r, \varphi, t) = tH^{(1)}(r, \varphi, t), \tag{59}$$

which demonstrates secular growth of the continuous-spectrum part of the perturbation. In this case, there are no specific parameters which can be varied to suppress the corresponding resonances, and the result is, in general, the destruction of the initial vortex. As an example, consider a vortex lying on the interface between two homogeneous fluids with densities $\rho_0 - \Delta\rho$ (upper) and ρ_0 (lower) so that the small parameter is $\mu = O(\Delta\rho/\rho_0)$. For simplicity, in this section we set the vortex radius $a = 1$ and vorticity $\omega = 1$. In this case, the solution (20) permits a relatively simple analysis in all limiting cases. If, for example, we are interested in the behaviour of flow perturbations outside the vortex ($r > 1$), the expression for the first azimuthal harmonic of this solution ($m = 1$) can be reduced to the form

$$\psi_1^{(1)}(r > 1, t) \approx \left(\frac{\sqrt{t}}{2r} - \frac{r}{4\sqrt{t}}\right)\int_0^{t/r^2} \tau^{-1/2}\mathrm{e}^{\mathrm{i}\tau}\mathrm{d}\tau + \frac{\mathrm{e}^{\mathrm{i}t/r^2}}{2\mathrm{i}} + \frac{\sqrt{t}}{r}\int_0^t \tau^{-1/2}\mathrm{e}^{\mathrm{i}\tau}\mathrm{d}\tau. \tag{60}$$

The last term in the right-hand part of this expression describes the translational mode of the discrete spectrum, whereas the previous terms correspond to the continuous spectrum. Note also that the continuous part is of a self-similar nature: it depends only on the combination t/r^2. Consider first the initial stage of the process when $t \ll 1$. This means that the angular particle shift in the whole vortex is still small compared to the complete rotation cycle. The perturbation at this stage is

$$\psi_1^{(1)} \sim \frac{\Delta\rho t}{2\rho_0 r} + O\left(\frac{t}{r^2}\right). \tag{61}$$

The main term here describes the vortex translation as a whole in the spirit of the general formula (28); indeed, the full solution can be represented in the form $\psi \simeq \Psi[(x + (\Delta\rho/2\rho_0)t)^2 + y^2]$. Evidently, this translation is due to the action of a vortex sheet which arises at the interface and creates a velocity parallel to the interface at the site of the vortex (cf. the simple case of vortex interaction with a hard surface). However, at this stage the displacement is much smaller than the vortex radius. In the opposite case, when $t \gg 1$ and the vortex has already performed many rotations, an inhomogeneity of particle motions is important. For the peripheral region where $r \gg \sqrt{t}$ and the fluid particle displacement is still small compared to their full rotation cycle, we have

$$\psi_1^{(1)} \sim \frac{\Delta\rho\sqrt{t}}{2\rho_0 r} + O\left(\frac{t}{r^2}\right), \tag{62}$$

which corresponds to the translational drift, which again is small compared with r.

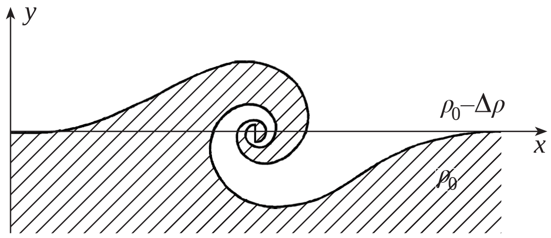

FIGURE 2. 'Winding' of a density interface around a vortex core.

For $r \ll \sqrt{t}$ where the particles have done many rotation cycles, the solution is

$$\psi_1^{(1)} \sim \frac{\Delta\rho}{\rho_0}\left(\frac{\sqrt{t}}{4r} + \frac{e^{it/r^2}}{2i}\right). \tag{63}$$

The first term here again describes the drift, and this term seems to dominate for large t. However, the physical motion is described by the particle velocities, which are derivatives of the stream function. In the expression for the azimuthal velocity ($v_\varphi \sim \partial_r\psi$), the last term is proportional to t, and it prevails over the previous one for large t. This oscillatory term describes the process of bending and even 'winding' of the interface (in the absence of gravity) around the vortex (figure 2), rather than the vortex drift as a whole. Such a winding is bound to result in the vortex destruction before it can drift on a distance exceeding its own radius. As will be shown below, the situation is different for a vortex pair in a stratified fluid. It is known from experiments and numerical calculations that such a pair can drift as a whole at sufficiently long distances. In what follows we shall be considering vortex pairs.

6. Dynamics of a disturbed vortex pair

6.1. *Equations for a pair of Rankine vortices*

The destructive role of resonances arising in the continuous spectrum depends on the spatial structure of the corresponding modes. In the case of a single vortex, the vorticity perturbations are moved around with the circular streamlines of the basic eddy (figure 3*a*). This results in cumulation and secular growth of perturbations and eventually in the destruction of the initial structure. The situation with a vortex pair is different. In this case only a part of the continuum-spectrum modes remains localized around the vortex cores, whereas other modes transfer the perturbations away from the vortex along the infinite streamlines (figure 3*b*). These modes prevent the cumulation of perturbations and preserve the structural entity of vortices for a longer time, as in the case for solitons in nonlinear wave systems. It should be noted that the unperturbed vortex dipole already has its own drift velocity (like a soliton) which is slowly varied by perturbations.

We consider here the dynamics of a vortex dipole which, in the zero approximation, is translated with a constant velocity. According to the general scheme developed in § 4, we suppose that the basic flow is a pair of vortices with equal but opposite signed vorticities (vortex dipole), and the coordinates of vortex centres are slowly varying functions of time. Then we obtain equation (42), in which, in the first approximation,

$$H^{(1)} = \sum_i \dot{\boldsymbol{R}}_i \nabla\Delta\Psi(r_i) - \sum_{i\neq j} J(\Psi(r_i)\Delta\Psi(r_j)) + H\left(\sum_i \Psi(r_i), \boldsymbol{r}, t\right). \tag{64}$$

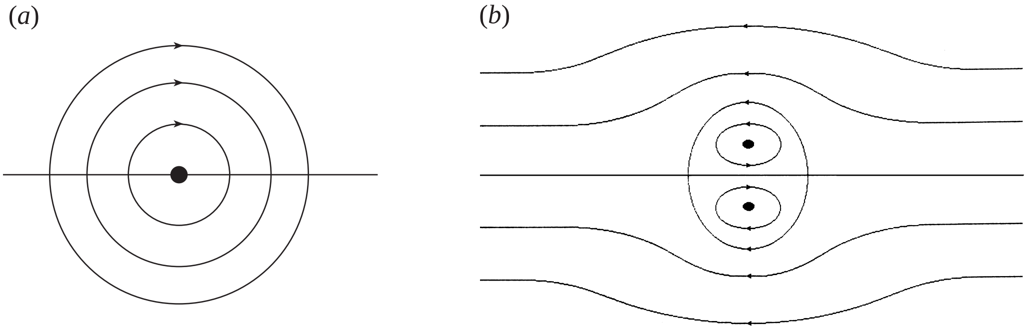

FIGURE 3. Streamlines carrying the vorticity perturbation: (*a*) for a single vortex; (*b*) for a vortex dipole.

The first two sums in this expression describe a superposition of translational eigenmodes for the respective eddies, so that this part of the expression corresponds to the discrete spectrum in the representation (46). As a result, for a distributed vortical component $v^{(1)}$ of the flow, we obtain the equation

$$\partial_t v^{(1)} + J\left(\sum_i \Psi(r_i), v^{(1)}\right) = H\left(\sum_i \Psi(r_i), \boldsymbol{r}, t\right), \tag{65}$$

and the expressions for the amplitudes of translational ($m = 1$) discrete-spectrum modes in (46) are modified as follows:

$$C_{1i}^{(1)}(t) = \frac{\omega_i}{4m}\int_0^t \mathrm{d}t'\left[a_i\left(-\dot{X}_i + \mathrm{i}\dot{Y}_i - \mathrm{i}\sum_{j\neq i}\frac{\omega_j a_j^2}{R_{ij}}\mathrm{e}^{-\mathrm{i}\varphi_{ij}}\right) + \mathrm{i}\int_0^\infty \mathrm{d}r_i f_1(r_i) r_i v_1^{(1)}(r_i, t)\right]. \tag{66}$$

Setting $C_{1j}^{(1)} = 0$, we obtain motion equations for the vortex centres. Considering that $\int_0^\infty f_1(r_i) v_1^{(1)}(r_i, t) r_i \mathrm{d}r_i = \int_0^\infty (\partial_y \Psi - \mathrm{i}\partial_x \Psi) v^{(1)}(\boldsymbol{r}_i, t)\mathrm{d}\boldsymbol{r}$, where $\mathrm{d}\boldsymbol{r} = \mathrm{d}x\,\mathrm{d}y$ (cf. (14)), these equations can be written in the form

$$\boldsymbol{\Omega} \times \frac{\mathrm{d}\boldsymbol{R}_{1,2}}{\mathrm{d}t} = \frac{\Omega^2}{\pi}\frac{\boldsymbol{R}}{R^2} - \pi\int_0^\infty \nabla\Psi(|\boldsymbol{r} - \boldsymbol{R}_{1,2}|)\cdot v^{(1)}(\boldsymbol{r}, t)\mathrm{d}\tilde{\boldsymbol{r}}, \tag{67}$$

where $\boldsymbol{\Omega}$ is the integral vorticity vector ($\boldsymbol{\Omega} = \int \Delta\Psi \mathrm{d}\boldsymbol{r}$) for a single vortex; in this case, $\Omega_1 = -\Omega_2 = \Omega$, $\boldsymbol{R} = \boldsymbol{R}_1 - \boldsymbol{R}_2$, so that $R = |\boldsymbol{R}|$ is the distance between the vortex centres. In view of the description of the vortex dipole motion as a whole, it is reasonable to rewrite (67) in variables $\boldsymbol{R}_0 = \frac{1}{2}(\boldsymbol{R}_1 + \boldsymbol{R}_2)$ so that $\boldsymbol{R}_0$ is a coordinate of the mass centre. This yields

$$\boldsymbol{\Omega} \times \frac{\mathrm{d}\boldsymbol{R}_0}{\mathrm{d}t} = \frac{\Omega^2\boldsymbol{R}}{\pi R^2} - \pi\int \nabla(\Psi(\boldsymbol{r}_1) - \Psi(\boldsymbol{r}_2))v^{(1)}\,\mathrm{d}\boldsymbol{r}, \tag{68a}$$

$$\boldsymbol{\Omega} \times \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = \pi\int \nabla\Psi(\boldsymbol{r})v^{(1)}\,\mathrm{d}\boldsymbol{r}. \tag{68b}$$

Here and below, $\Psi(\boldsymbol{r})$ denotes the sum $\Psi(\boldsymbol{r}_1) + \Psi(\boldsymbol{r}_2)$, which is a stream function for the unperturbed vortex pair. Equations (68) together with (65) form a closed system. In fact equation (68*a*) (without the integral term) gives a kinematic relation for an unperturbed vortex pair's coordinates and its velocity, and (68*b*) describes the variation of the fluid momentum due to perturbation.

The solution of the transfer equation (65) for $v^{(1)}$ can be constructed with the characteristics:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \Psi_y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \Psi_x, \qquad \frac{\mathrm{d}v^{(1)}}{\mathrm{d}t} = H(\Psi(\boldsymbol{r}), \boldsymbol{r}, t). \tag{69}$$

It has the form

$$v^{(1)}(\boldsymbol{r}, t) = \int H(\Psi(\boldsymbol{r}), \boldsymbol{r}, t)\,\mathrm{d}t, \tag{70}$$

where the integral is taken along the characteristic lines. Substituting (70) into (68), we come, in general, to a rather cumbersome system of integral-differential equations which can be, however, considerably simplified in some important cases.

6.2. *Dynamics of a vortex pair in a smoothly stratified fluid*

A visual geometric interpretation of the solutions for the transport equation permits one to single out some constructive approximations. We consider here the motion of a vortex pair in a density-stratified fluid. The first-order equation (52) for $v^{(1)}$ has the form

$$\partial_t v^{(1)} + J\left(\Psi(\boldsymbol{r}), v^{(1)}\right) = H = \frac{\nabla\rho^{(1)}}{\rho} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\nabla\Psi(\boldsymbol{r}), \tag{71}$$

where the density perturbation $\rho^{(1)}$ satisfies the inhomogeneous transfer equation (53). We suppose first that the characteristic scale of the initial density distribution $\rho^{(1)}(\boldsymbol{r}, t=0)$ is much larger than that of the vortex pair (such as distance between the vortex centres). As long as this relationship between the scales is preserved in the course of evolution, the function $\nabla\rho$ can be extracted from the integrand in (70), where H is defined by (71) to give the following solution for $v^{(1)}$,

$$v^{(1)} = \frac{\nabla\rho(\boldsymbol{R}_0) \cdot \nabla\Psi(\boldsymbol{r}, \boldsymbol{R}_0, \boldsymbol{R})}{\rho(\boldsymbol{R}_0)}. \tag{72}$$

Such a 'frozen density' approximation, when density gradient is considered locally constant at the scale of the solution, is somewhat analogous to the one used for the description of soliton dynamics in smoothly inhomogeneous media. Then, substitution of (72) into (68) reduces the latter to a system of ordinary differential equations:

$$\boldsymbol{\Omega} \times \frac{\mathrm{d}\boldsymbol{R}_0}{\mathrm{d}t} = \frac{\Omega^2 \boldsymbol{R}}{\pi R^2}, \qquad \boldsymbol{\Omega} \times \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = \pi \int \left(\frac{\nabla\rho}{\rho} \cdot \nabla\Psi\right) \nabla\Psi\,\mathrm{d}\boldsymbol{r}, \tag{73}$$

where $\nabla\rho/\rho$ is considered a constant. In the first of these equations a term proportional to $\nabla\rho$ is omitted: due to the aforementioned scale difference, it has a higher order of smallness. The system (73) admits a detailed analytical study in the particular case of plane stratification, when $\rho^{(1)}$ depends on one coordinate y. Because the distance between the vortex centres significantly exceeds their sizes ($R \gg a$), the integrals in (73) can be substituted by their approximate expressions:

$$\left.\begin{aligned} \int \Psi_x \Psi_y\,\mathrm{d}\boldsymbol{r} &= 2\pi\omega^2 a^4 \left[\frac{R_x R_y}{R^2} + O\left(\frac{a^2}{R^2}\right)\right], \\ \int \Psi_y^2\,\mathrm{d}\boldsymbol{r} &= \pi\omega^2 a^4 \left[\frac{(R_y^2 - R_x^2 + R^2 \ln(4R^2/a^2)}{R^2} + O\left(\frac{a^2}{R^2}\right)\right]. \end{aligned}\right\} \tag{74}$$

As a result, the system (73) becomes

$$\frac{\mathrm{d}X_0}{\mathrm{d}t} = -\frac{R_y}{R^2}, \qquad \frac{\mathrm{d}Y_0}{\mathrm{d}t} = \frac{R_x}{R^2}, \tag{75a,b}$$

$$\frac{\mathrm{d}R_x}{\mathrm{d}t} = \pi \frac{R_y^2 - R_x^2 + R^2 \ln R^2}{R^2} \frac{\rho^{(1)\prime}(Y_0)}{\rho(Y_0)}, \tag{75c}$$

$$\frac{\mathrm{d}R_y}{\mathrm{d}t} = -\pi \frac{2R_x R_y}{R^2} \frac{\rho^{(1)\prime}(Y_0)}{\rho(Y_0)}, \tag{75d}$$

$$\boldsymbol{R}_0 = (X_0, Y_0), \qquad \boldsymbol{R} = (R_x, R_y), \tag{75e,f}$$

where X_0, Y_0, R, and $R_{x,y}$ are normalized by a, and t by Ω^{-1}. Only (75*b*–*d*) in this system are independent. It is interesting that these equations have two analytical integrals. From (75*c,d*) it follows that

$$R_y [\ln R^2 - 1] = c = \text{const}, \tag{76}$$

and (75*b,d*) give

$$\ln (R_y/R_{y0}) = \pi \ln \left[\rho(Y_0)/\rho(Y_{00})\right], \tag{77}$$

where R_{y0} and Y_{00} are integration constants which are chosen to be the initial values of R_y and of the height of the pair, respectively. As a result, a general solution of (75) can be reduced to quadratures. A general description of the trajectories of the eddies in the pair can be obtained from (76) and (77). Indeed, the presence of two integrals in a third-order system means the possibility of separating its phase space into the phase trajectories lying at cylindrical surfaces oriented along the Y_0-axis, and the contours in the (R_x, R_y)-plane given by the integral (76). Note that the system (76) and (77) is applicable only if $R \gg 1$; this means that finite trajectories in which R_x and R_y are bounded simultaneously (for them $R \leqslant \sqrt{e}$), are excluded from consideration. Figure 4 shows the trajectories corresponding to different initial conditions (different c). For all trajectories which are within the limits of present consideration, $R_x \in (-\infty, \infty)$ and $R_y \in [0, R_{y\,\max}]$. This means that, for a monotonic $\delta\rho(y)$, a typical trajectory in the (X_0, Y_0)-plane corresponds to motion with a reflection (figure 5). Due to invariance of the equations to the change $t \to -t, R_x \to -R_x$, such a trajectory is symmetric with respect to the vertical axis y. The coordinates of the reflection point, (R_{xr}, R_{yr}), are defined by

$$R_{xr} = 0, \qquad R_{yr}(\ln R_{yr}^2 - 1) = c\,; \tag{78}$$

in particular, its vertical coordinate satisfies the relation

$$\pi \ln [\rho(Y_{0,r})/\rho(Y_{00})] = \ln (R_{yr}/R_{y0}). \tag{79}$$

These parameters do not depend on the vorticity value in the pair but only on the geometrical parameters. Note that the pair velocity tends to zero at $t \to \infty$ only if $\rho(y \to \infty) = 0$. The time interval at which the solutions to equations (73) are applicable is limited, but it radically exceeds that for a single vortex: the pair can be translated as a whole to a distance of order the characteristic scale of the density variation. Suppose that the characteristic spatial variation scale L for the function $\rho(t = 0, r)$ is of order μ^{-2}, where $\mu \sim a/R$ is the small parameter. Then, as follows from (73), during the time taken to travel a distance L by the pair, its size changes by its own order. A fluid mass entrained into the pair (a 'trapping' area) or leaving it is also changed by its order. As a result, during this time, a region with sharper

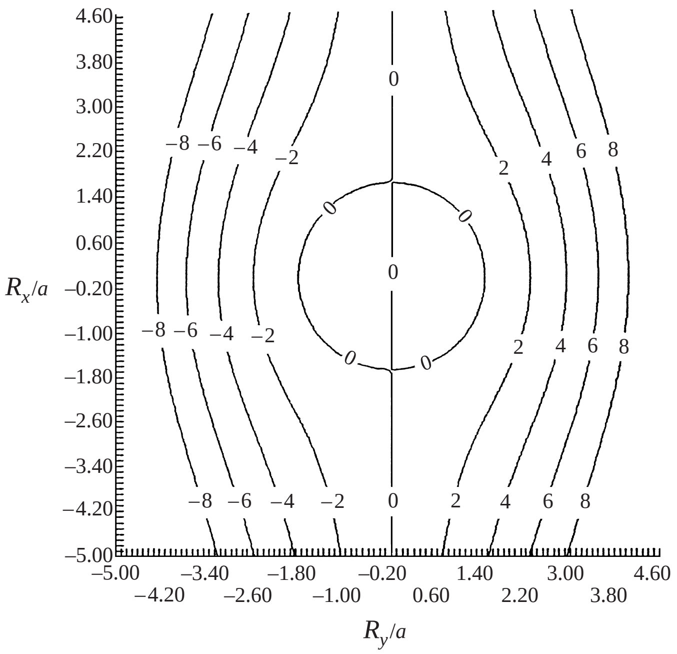

FIGURE 4. Projections of phase trajectories of the system (75) on the plane of variables R_x, R_y. Each curve corresponds to a constant c.

(with a scale of order R instead of L) density variations is formed,† so that the solution (72) for $v^{(1)}$ (hence a solution to (73)) is no longer valid, and a more general integral-differential system (68) and (70) should be used which takes into account the detailed structure of density and vorticity perturbations.

6.3. *Dynamics of a vortex pair in a fluid with a density jump*

As we have already seen, a single vortex at the interface is destined to be destroyed in a time short compared to that of its drift a distance larger than its own size. It can be shown that this is also true for a vortex moving far from the interface. Indeed, in the latter case both terms in (63) will acquire factors of order $(a/y)^2$, where y is the distance from the interface, and the ratio between the corresponding terms remains the same in order. For a vortex pair, the situation is again more stable; moreover, in some cases a 'frozen density' approximation can still work. One such case is that of a pair moving sufficiently far from the area of sharp density gradients. This implies that the fluid particle displacements and their velocities near the boundary, as well as the perturbations of the vortex pair velocities, are small enough. As a result, it is possible to neglect the convective terms $J(\Psi, \rho)$ and $J(\Psi, v^{(1)})$ in equations (55) and (71) for ρ and $v^{(1)}$, respectively. In this case, the induced vorticity can be found from the simplified equation (71):

$$\partial_t v^{(1)} = \frac{\nabla\rho \cdot \partial_t \nabla\Psi}{\rho}, \tag{80}$$

† Relatively large density gradients also occur without a change of the trapping area, if $\dot{R} = 0$ (but $\dot{\boldsymbol{R}} \neq 0$): here, the fluid particles with different densities are localized outside the trapped area.

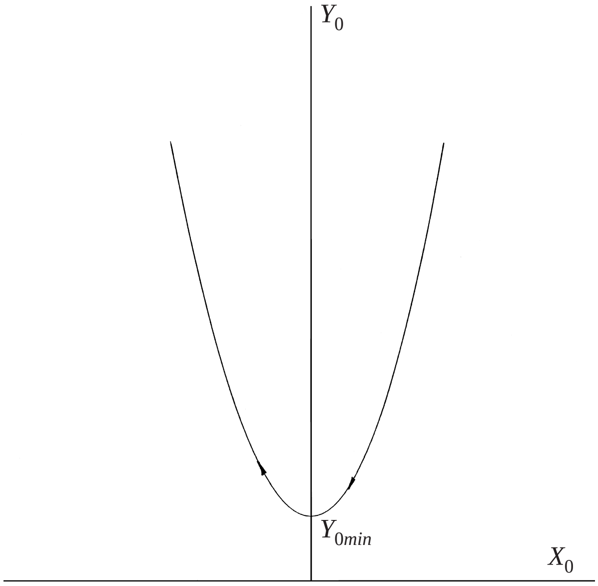

FIGURE 5. A typical path of the vortex pair in the (X_0, Y_0)-plane for a monotonic function $\rho(y)$.

having a solution

$$v^{(1)} = \frac{\nabla\rho(\boldsymbol{r})\cdot\nabla\Psi(\boldsymbol{r},t)}{\rho(\boldsymbol{r})}. \tag{81}$$

Substitution of (81) into the system (68) leads to the following equations for the parameters of the pair:

$$\left.\begin{aligned} \boldsymbol{\Omega}\times\frac{\mathrm{d}\boldsymbol{R}_0}{\mathrm{d}t} &= \frac{\Omega^2}{\pi}\frac{\boldsymbol{R}}{R^2}, \\ \boldsymbol{\Omega}\times\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} &= \int\left(\frac{\nabla\rho(\boldsymbol{r})\cdot\nabla\Psi(\boldsymbol{r},t)}{\rho(\boldsymbol{r})}\right)\nabla\Psi(\boldsymbol{r},t)\,\mathrm{d}\boldsymbol{r}. \end{aligned}\right\} \tag{82}$$

Now the density does not vary slowly, and the factor $(\nabla\rho/\rho)$ cannot be treated as a constant. On the other hand, the remoteness of the pair from the boundary makes it possible to omit in (68*a*) the last term which is proportional to $\nabla\rho$. For the same reason, the components of the vector $\nabla\Psi$ can be substituted by their asymptotic expressions corresponding to the flow far from the pair:

$$(\Psi^{(0)})_x = -\Omega\frac{[(y-Y_0)^2-(x-X_0)^2]R_x - 2(x-X_0)(y-Y_0)R_y}{[(x-X_0)^2+(y-Y_0)^2]^2}, \tag{83a}$$

$$(\Psi^{(0)})_y = \frac{[(y-Y_0)^2-(x-X_0)^2]R_y + 2(x-X_0)(y-Y_0)R_x}{[(x-X_0)^2+(y-Y_0)^2]^2}, \tag{83b}$$

where $\boldsymbol{R}_0 = \{X_0, Y_0\}$ and $\boldsymbol{R} = \{R_x, R_y\}$. Consider a pair motion in the presence of an interface $(y = 0)$ between fluids of different but close densities ρ_1 and ρ_2:

$$\rho(y) = \begin{cases} \rho_0 - \delta\rho, & y > 0, \\ \rho_0 + \delta\rho, & y < 0, \end{cases} \qquad \delta\rho > 0, \delta\rho = \rho_2 - \rho_1 \ll \rho_0. \tag{84}$$

The pair moves in such a way that the distance R_0 between vortices remains much less than the distance r of the pair centre from the interface. The system (73) now reads

$$\frac{\mathrm{d}X_0}{\mathrm{d}t} = -\frac{R_y}{R^2} \tag{85}$$

$$\frac{\mathrm{d}Y_0}{\mathrm{d}t} = \frac{R_x}{R^2} \tag{86}$$

$$\frac{\mathrm{d}R_x}{\mathrm{d}t} = \frac{\pi}{2}\frac{\delta\rho}{\rho_0}\frac{R^2}{Y_0^3} \tag{87}$$

$$\frac{\mathrm{d}R_y}{\mathrm{d}t} = 0. \tag{88}$$

Let us analyse this system. First, the vertical component R_y of the vector $\boldsymbol{R}$ connecting the vortex centres is conserved. This follows just from the translational symmetry along the x-axis: the change of the full x-component of fluid momentum (that includes the momentum of the dipole and the vortex sheet at the interface) is equal to zero in this example. Note that in the case of smooth stratification considered above this momentum component is non-zero, whence R_y varies in time. Then the second-order system (86) and (87) is separated. It can be represented on the phase plane of variables R_x and Y_0 (figure 6). The equation of phase trajectories has the form

$$Y_0^2 = \frac{\pi\delta\rho}{2\rho_0}(\operatorname{sign} Y_0)\frac{R_x^2 + R_{y0}^2}{1 + C(R_x^2 + R_{y0}^2)}, \tag{89}$$

where $C \in (-\infty, \infty)$ is an integration constant. Of qualitative importance is the phase trajectory $Y_0 = 0$ which corresponds to $C \to \pm\infty$. The existence of this trajectory means that the pair never intersects the interface $Y_0 = 0$. Still, the character of motion depends on the initial position of the pair. If it moves in the half-space of smaller density ($y > 0$, $C \in (-R_{y0}^{-2}, \infty)$), all phase trajectories are reflected from the boundary, with a minimal distance from the interface equalling

$$(Y_0^2)_{min} = \frac{\pi\delta\rho}{2\rho_0}\frac{R_{y0}^2}{1 + CR_{y0}^2}. \tag{90}$$

The trajectories are symmetric: the pair is approaching the density interface and moving away from it. However, in this case only part of the trajectories remains far from the boundary: only those for which $C \in (-R_{y0}^{-2}, 0)$. In this range, the constant C can be redefined as $C = -(R_{x\infty}^2 + R_{y0}^2)^{-1}$, where $R_{x\infty}^2$ is an asymptotic (at $t \to \pm\infty$) value of R_x^2. Substituting C into (90), we conclude that at $\delta\rho \ll \rho$, the conditions of applicability of the system (85)–(88) are fulfilled for the entire motion, only for small angles of incidence of the pair onto the interface ($R^2 \ll \rho_0 R_{y0}^2/\delta\rho$). If the pair is situated in the region with larger density ($y < 0, C(-\infty, 0)$), it tends to be attracted to the interface. There still exists a range of initial conditions ($R_x < 0$, $C \in (-R_{y0}^{-2}, 0)$) under which the pair does not approach the boundary and eventually returns to infinity. In this case, however, no complete phase trajectories are compatible with the condition of the applicability of equations (85)–(88): all trajectories have one or both asymptotes with finite values of Y_0 at $R \to \pm\infty$ (figure 6).† Nonetheless, for the parts of the trajectories with relatively small values of R and relatively large Y, we have an adequate description. This analysis can be readily extended to the case of two

† A similar behaviour of phase trajectories takes place in the region of reflection, if $C \in (0, \infty)$.

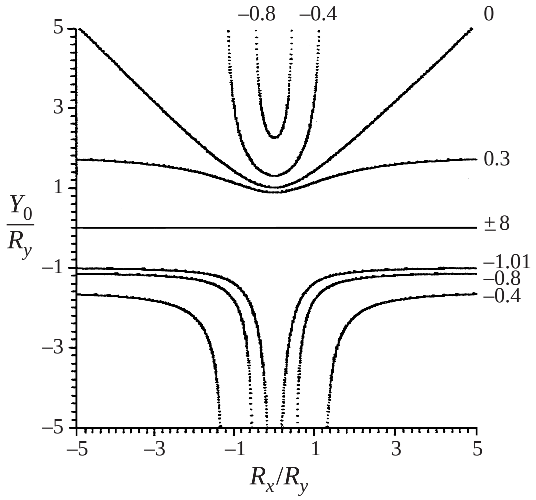

FIGURE 6. Phase portrait of the system (86), (87) for a fixed R_y. The value of C (integration constant) for each curve is indicated.

boundaries; in this case the possibility of localization of the pair moving between two interfaces can be foreseen.

The effect of vortex pair reflection for $\Delta\rho > 0$ can be explained by the action of the vorticity induced at the interface; it leads to the additional drift of each vorticity patch in the pair along x, which is evidently stronger for the vortex closer to the interface. Because the distance R_y between the patches is conserved, the effect is the rotation of the dipole and eventual change of the vertical component of the basic dipole velocity. Qualitatively, the same is true for the case of smooth stratification considered above.

7. Conclusions

Here, we have developed an asymptotic perturbation theory for description of the dynamics of localized vortices and vortex systems interacting in the presence of external perturbations such as density variations. Basic ideas of the method are somewhat analogous to those known in soliton theory, but due to the non-wave character of the motion, the hydrodynamical problem turns out to be much more complicated. In this paper, the scheme was implemented only for Rankine vortices (vorticity patches). For smooth vorticity distributions the temporal form of the Green function cannot, in general, be represented as a sum of terms corresponding to poles in the discrete and continuous spectra (cf. Fjortoft 1950 for plane flows). The relevant difficulty has already been mentioned in § 2.3 with regard to the logarithmic branching of functions $f_{1,2m}$. It may be supposed, however, that at least some of the above results will remain applicable to the general case of smooth vorticity profiles. In particular, as already mentioned, equations (68) which describe the vortex pair dynamics, have a visual physical interpretation: (68*a*) gives a kinematic relation for an unperturbed vortex pair's coordinates and its velocity, whereas (68*b*) describes the variation of the fluid momentum. It is natural to believe that similar relationships should remain valid for pairs of vortices of an arbitrary structure.

Besides the development of a general scheme, we also considered some specific processes such as the vortex pair reflection from the interface between two fluids with slightly different densities or from some level in a fluid with a smooth density gradient. It may be expected that asymptotic methods will be used more widely in hydrodynamics of vortices, to reach an analytical (or semi-analytical, or simplified numerical) description of vortex drift and their possible destruction in various geophysical situations and in laboratory experiments. We are planning to extend the theory in several directions by incorporating, in particular, gravity effects (arbitrary Froude numbers for a vortex rather than the large ones which were considered above) and rotation effects, which are of primary interest in geophysical hydrodynamics (see, e.g., McWilliams & Flierl 1979 and Zabusky & McWilliams 1982); the paper by Reznik & Dewar (1994) in which the β-effect was considered as a small perturbation to a symmetric barotropic vortex should be specially mentioned here. An important feature of these problems is the appearance of waves which, on the one hand, may additionally stabilize the vortex by carrying the perturbations away but, on the other, complicate the solution because the far-field motions must be matched to those in the vortex vicinity.

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REFERENCES

Abrashkin, A. A. 1987 Theory of interaction between two plane vortices in a perfect fluid. *Fluid Dyn.* **22,** 53–59.

Batchelor, G. K. 1967 *An Introduction to Fluid Dynamics*. Cambridge University Press.

Dahm, W. J. A., Sheil C. M. & Tryggvason, G. 1989 Dynamics of vortex interaction with a density interface. *J. Fluid Mech.* **205**, 1–43.

Deem, G. & Zabusky, N. J. 1978 Stationary 'V-states', interactions, recurrence, and breaking. In *Solitons in Action* (ed. K. Lonngren & A. Scott), pp. 277–293. Academic.

Dikii, L. A. 1960 The stability of plane-parallel flows of an ideal fluid. *Sov. Phys.-Dokl.* **5**, 1179–1182.

Fjortoft, R. 1950 Application of integral theorems in deriving criteria of stability for laminar flows and for the baroclinic circular vortex. *Geofys. Publ. Oslo* **17,** 1–52.

Gorshkov, K. A. & Ostrovsky, L. A. 1981 Interaction of solitons in non-integrable systems. *Physica* **3D**, 428–438.

Grimshaw, R. H. J. 1979 Slowly varying solitary waves. I. Korteweg–de Vries equation. *Proc. R. Soc. Lond.* A **368**, 359–375.

Hill, F. M. 1975 A numerical study of the descent of a vortex pair in a stratified atmosphere. *J. Fluid Mech.* **71,** 1–13.

Keener, J. R. & McLaughlin, D. W. 1977 Soliton under perturbations. *Phys. Rev.* A **16**, 777–790.

Kurgansky, M. V. 1986 Motion of a vortex pair beneath the surface of a heavy fluid. *Fluid Dyn.* **21,** 184–186.

Lamb, H. 1982 *Hydrodynamics*, 6th Edn. Cambridge University Press.

Llewellyn Smith, S. G. 1995 The influence of circulation on the stability of vortices to one-mode disturbance. *Proc. R. Soc. Lond.* A **451**, 747–755.

Manakov, S. V. & Shchur, L. N. 1983 Stochastic aspect of two-particle scattering. *JETP Lett.* **37**, 54–57.

McWilliams, J. C. & Flierl, G. R. 1979 On the evolution of isolated, nonlinear vortices. *J. Phys. Oceanogr.* **9**, 1155–1182.

Melander, M. V., Styezek, A. S. & Zabusky, L. J. 1972 Elliptical desingularized vortex model for two-dimensional Euler equation. *Phys. Rev. Lett.* **11**, 107.

Reznik, G. M. & Dewar, W. K. 1994 An analytical theory of distributed axisymmetric barotropic vortices on the β-plane. *J. Fluid Mech.* **269**, 301–321.

Saffman, P. 1979 The approach of a vortex pair to a plane surface in inviscid fluid. *J. Fluid Mech.* **92**, 497–503.

SMITH, G. B. & MONTGOMERY, M. T. 1995 Vortex axisymmetrization: Dependence on azimuthal wave-number or asymmetric radial structure changed. *Q. J. Met. Soc.* **121**, 1615–1651.

SMITH, R. A. & ROSENBLUTH, M. N. 1991 *Phys. Rev. Lett.* **64**, 649–652.

TURNER, J. 1973 *Buoyancy Effects in Fluids.* Cambridge University Press.

ZABUSKY, N. J. & MCWILLIAMS, J. C. 1982 A modulated point-vortex model for geostrophic, β-plane dynamics. *Phys. Fluids* **25**, 2175–2183.